Quantum Computing

Unit 2: Linear Algebra and the Dirac Notation

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Course Page:

https://www.gcjana.in/courses/shardauniversity/2501/quantum-computing/

Outline

Learning Outcomes of Unit 2

After this unit, a student will be able to...

1. **Use** Dirac (bra–ket) notation for vectors, dual vectors, inner and outer products.
2. **Define** a Hilbert space and its inner product; represent kets and bras as column / row vectors.
3. **Manipulate** operators and adjoints; identify *Hermitian*, *unitary*, *normal*, *positive* and *projection* operators.
4. **State** and apply the *spectral theorem*; compute spectral decompositions.
5. **Define** functions of operators, including the operator exponential.
6. **Build** tensor-product spaces and compute Kronecker products.
7. **State** the *Schmidt decomposition* theorem and use the Schmidt rank to detect entanglement.

Mapping to the Syllabus

A. Dirac notation & Hilbert spaces, dual vectors, operators. **B.** Spectral theorem, functions of operators. **C.** Tensor products, Schmidt decomposition theorem.

Why Linear Algebra? (Bridge from Unit 1)

The single mathematical language of QC

In Unit 1 we saw states as vectors and gates as matrices. Unit 2 makes that precise with **Dirac notation** and the theory of operators.

- **States** $\leftrightarrow$ unit vectors $|\psi\rangle$ in a Hilbert space.
- **Measurements / observables** $\leftrightarrow$ *Hermitian* operators.
- **Evolution / gates** $\leftrightarrow$ *unitary* operators.
- **Composite systems** $\leftrightarrow$ *tensor products*.

Convention

We work in **finite-dimensional** complex spaces $\mathbb{C}^n$ (all a qubit register needs). There, "Hilbert space" just means $\mathbb{C}^n$ with the standard inner product.

Notation preview

$|\psi\rangle$ ket (column), $\langle\psi|$ bra (row), $\langle\phi|\psi\rangle$ inner product, $|\phi\rangle\langle\psi|$ outer product (an operator).

Complex Vector Spaces and Hilbert Spaces

Hilbert space $\mathcal{H}$

A **Hilbert space** is a complex vector space equipped with an *inner product* that is *complete* (every Cauchy sequence converges). In **finite dimension** completeness is automatic, so $\mathcal{H} \cong \mathbb{C}^n$.

Vector-space essentials

For $|\psi\rangle, |\phi\rangle \in \mathcal{H}$ and $a, b \in \mathbb{C}$:

- $a|\psi\rangle + b|\phi\rangle \in \mathcal{H}$ (closure),
- a zero vector $|0\rangle$ (*not* the qubit $|0\rangle$!),
- associativity, commutativity, distributivity.

Dimension & basis

An **orthonormal basis** $\{|0\rangle, \ldots, |n-1\rangle\}$ satisfies $\langle i|j\rangle = \delta_{ij}$. Every state expands uniquely:

$$|\psi\rangle = \sum_i c_i |i\rangle, \qquad c_i = \langle i|\psi\rangle \in \mathbb{C}.$$

A qubit lives in $\mathbb{C}^2$; n qubits in $\mathbb{C}^{2^n}$.

Kets — Vectors in Dirac Notation

Column-vector picture

Fixing an orthonormal basis, a ket is a column of amplitudes:

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \quad |0\rangle = \begin{pmatrix}1\\ 0\end{pmatrix}, \; |1\rangle = \begin{pmatrix}0\\ 1\end{pmatrix}.$$

- Amplitudes $\alpha, \beta \in \mathbb{C}$.
- A *physical state* is normalised: $|\alpha|^2 + |\beta|^2 = 1$.
- Global phase $e^{i\gamma}\,|\psi\rangle$ is physically irrelevant.

Useful named states

$$|+\rangle = \tfrac{1}{\sqrt{2}}\big(\,|0\rangle + |1\rangle\,\big), \quad |-\rangle = \tfrac{1}{\sqrt{2}}\big(\,|0\rangle - |1\rangle\,\big).$$

These form another orthonormal basis of $\mathbb{C}^2$.

Reminder

$|\cdot\rangle$ is just a *name*: $|\psi\rangle$, $|\uparrow\rangle$, $|01\rangle$ are all vectors.

The Inner Product

Definition

The inner product $\langle \phi | \psi \rangle \in \mathbb{C}$ satisfies, for all $a, b \in \mathbb{C}$:

- **Conjugate symmetry:** $\langle \phi | \psi \rangle = \langle \psi | \phi \rangle^*$.
- **Linear in the ket (2nd) slot:** $\langle \phi | \big(a \, |\psi_1\rangle + b \, |\psi_2\rangle \big) = a \langle \phi | \psi_1 \rangle + b \langle \phi | \psi_2 \rangle$.
- **Positive definite:** $\langle \psi | \psi \rangle \geq 0$, with equality iff $|\psi\rangle = 0$.

Norm and coordinates

$$\| \, |\psi\rangle \, \| = \sqrt{\langle \psi | \psi \rangle}, \qquad \langle \phi | \psi \rangle = \sum_i \phi_i^* \, \psi_i.$$

Physics convention

The first (bra) slot is *conjugate-linear*, the second (ket) slot is *linear*. Also Cauchy–Schwarz: $| \langle \phi | \psi \rangle | \leq \|\phi\| \, \|\psi\|$.

Dual Vectors — Bras

Bra = linear functional

A **bra** $\langle\psi|$ is a linear map $\mathcal{H} \to \mathbb{C}$, $|\chi\rangle \mapsto \langle\psi|\chi\rangle$. It is the *dual vector* of $|\psi\rangle$.

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \xrightarrow{\dagger} \langle\psi| = |\psi\rangle^\dagger = \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix}.$$

- The ket $\to$ bra map is **antilinear**: $a\,|\psi\rangle + b\,|\phi\rangle \;\to\; a^*\,\langle\psi| + b^*\,\langle\phi|$.

Riesz representation theorem

Every linear functional on a (finite-dim.) Hilbert space equals $\langle\psi|\cdot\rangle$ for a **unique** $|\psi\rangle$. So bras and kets are in one-to-one correspondence.

Read it as "bra-c-ket"

$\underbrace{\langle\phi|}_{\text{bra}}\ \underbrace{|\psi\rangle}_{\text{ket}} = \langle\phi|\psi\rangle$ — row $\times$ column = a number.

Bra–Ket Mechanics

Orthonormality

$$\langle 0|0\rangle = \langle 1|1\rangle = 1, \qquad \langle 0|1\rangle = \langle 1|0\rangle = 0,$$

i.e. $\langle i|j\rangle = \delta_{ij}$.

Example

With $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$:

$$\langle\psi|\psi\rangle = \left|\frac{1}{\sqrt{2}}\right|^2 + \left|\frac{i}{\sqrt{2}}\right|^2 = \frac{1}{2} + \frac{1}{2} = 1. \checkmark$$

Completeness (resolution of identity)

For any orthonormal basis $\{|i\rangle\}$,

$$\sum_i |i\rangle\langle i| = I.$$

Why it's the workhorse

Insert $I = \sum_i |i\rangle\langle i|$ anywhere to expand in a basis:

$$|\psi\rangle = I\,|\psi\rangle = \sum_i |i\rangle\,\langle i|\psi\rangle = \sum_i c_i\,|i\rangle\,.$$

Outer Products and Projectors

Outer product = operator

$|\phi\rangle\langle\psi|$ maps kets to kets:

$$\big(\,|\phi\rangle\langle\psi|\,\big)\,|\chi\rangle = \langle\psi|\chi\rangle\ |\phi\rangle\,.$$

As a matrix it is a column $\times$ row.

Projector onto $|\psi\rangle$ (normalised)

$$P = |\psi\rangle\langle\psi|\,, \qquad P^2 = P, \qquad P^\dagger = P.$$

Check: $P^2 = |\psi\rangle\,\underbrace{\langle\psi|\psi\rangle}_{=1}\,\langle\psi| = P.$

Example matrices

$$|0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad |0\rangle\langle 1| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Note

$\big(\,|\phi\rangle\langle\psi|\,\big)^\dagger = |\psi\rangle\langle\phi|$ — taking the adjoint *swaps* bra and ket.

Linear Operators and the Adjoint

Operators

A linear operator $A : \mathcal{H} \to \mathcal{H}$ acts as $|\psi\rangle \mapsto A|\psi\rangle$. Its matrix entries are

$$A_{ij} = \langle i| A |j\rangle .$$

Composition of operators = matrix multiplication.

The adjoint $A^\dagger$

Defined by $\langle\phi|A|\psi\rangle^* = \langle\psi|A^\dagger|\phi\rangle$ for all ϕ, ψ. In coordinates $A^\dagger = (A^*)^\top$ (**conjugate transpose**).

Algebra of the dagger

$$(A^\dagger)^\dagger = A, \quad (AB)^\dagger = B^\dagger A^\dagger,$$

$$(aA + bB)^\dagger = a^* A^\dagger + b^* B^\dagger.$$

$$|\psi\rangle \longrightarrow \boxed{A} \longrightarrow A|\psi\rangle$$

Figure: an operator transforms a state.

Special Classes of Operators

Type	Defining condition	Role in QC
Hermitian (self-adjoint)	$A^\dagger = A$	observables (real eigenvalues)
Unitary	$U^\dagger U = UU^\dagger = I$	gates / evolution (norm-preserving)
Normal	$A^\dagger A = AA^\dagger$	exactly the unitarily-diagonalisable ones
Positive (semidefinite)	$\langle\psi\|A\|\psi\rangle \geq 0 \ \forall\psi$	density operators
Projector	$P^2 = P = P^\dagger$	measurement outcomes

Unitaries preserve inner products

$$\langle U\phi|U\psi\rangle = \langle\phi|\, U^\dagger U \,|\psi\rangle = \langle\phi|\psi\rangle \,.$$

Containments

Hermitian $\subset$ normal and unitary $\subset$ normal; every positive operator is Hermitian.

Worked Example — Adjoint, Hermitian, Unitary

Is A Hermitian?

$$A = \begin{pmatrix} 2 & 1-i \\ 1+i & 3 \end{pmatrix}$$

$A^\dagger = (A^*)^\top$: conjugate $\Rightarrow \begin{pmatrix} 2 & 1+i \\ 1-i & 3 \end{pmatrix}$,

transpose $\Rightarrow \begin{pmatrix} 2 & 1-i \\ 1+i & 3 \end{pmatrix} = A.$

$$\boxed{A^\dagger = A \;\Rightarrow\; \text{Hermitian}}$$

Is the Hadamard H unitary?

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad H^\dagger = H.$$

$$H^\dagger H = H^2 = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = I.$$

$$\boxed{H^\dagger H = I \;\Rightarrow\; \text{unitary (and Hermitian)}}$$

Quick Check — Round 1 (Dirac Notation & Operators)

[Q1]

Write the bra corresponding to $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

[Q2]

Compute $\langle 0|1\rangle$ and $\langle +|+\rangle$, where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

[Q3]

State the defining relation of the adjoint $A^\dagger$ and give $(AB)^\dagger$.

[Q4] True/False

"Every unitary operator is normal."

Think for a minute — answers on the next slide.

Answers — Round 1

[A1]

$\langle\psi| = \alpha^* \langle 0| + \beta^* \langle 1| = \begin{pmatrix}\alpha^* & \beta^*\end{pmatrix}$ (the ket→bra map conjugates the amplitudes).

[A2]

$\langle 0|1\rangle = \mathbf{0}$ (orthogonal); $\langle +|+\rangle = \frac{1}{2}(\langle 0|0\rangle + \langle 0|1\rangle + \langle 1|0\rangle + \langle 1|1\rangle) = \frac{1}{2}(1+0+0+1) = \mathbf{1}$.

[A3]

$\langle\phi|A|\psi\rangle^* = \langle\psi|A^\dagger|\phi\rangle$ for all ϕ, ψ; and $(AB)^\dagger = B^\dagger A^\dagger$.

[A4]

True. $U^\dagger U = UU^\dagger = I$ gives $U^\dagger U = UU^\dagger$, which is the definition of *normal*.

Eigenvalues and Eigenvectors

Definition

$|v\rangle \neq 0$ is an **eigenvector** of A with **eigenvalue** λ if

$$A|v\rangle = \lambda|v\rangle .$$

Eigenvalues solve the characteristic equation $\det(A - \lambda I) = 0$.

Example — Pauli Z

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Z|0\rangle = |0\rangle , \; Z|1\rangle = -|1\rangle .$$

Eigenvalues ± 1 with eigenvectors $|0\rangle , |1\rangle$.

- The *eigenspace* of λ is $\{|v\rangle : A|v\rangle = \lambda|v\rangle\}$.
- *Degeneracy* $=$ dimension of an eigenspace > 1.

The Spectral Theorem

Statement (finite dimension)

Let A be a **normal** operator ($A^\dagger A = AA^\dagger$) on $\mathcal{H}$. Then A has an *orthonormal basis of eigenvectors* $\{|v_i\rangle\}$ with $A\,|v_i\rangle = \lambda_i\,|v_i\rangle$, and

$$\boxed{A = \sum_i \lambda_i\,|v_i\rangle\langle v_i|} \qquad \text{(spectral decomposition).}$$

Equivalently, there is a unitary U with $U^\dagger A U = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$.

Two important special cases

- A **Hermitian** $\Rightarrow$ all $\lambda_i \in \mathbb{R}$.
- A **unitary** $\Rightarrow$ all $|\lambda_i| = 1$.

Degenerate form

Grouping repeated eigenvalues, $A = \sum_\lambda \lambda\, P_\lambda$, with orthogonal projectors P_λ, $\sum_\lambda P_\lambda = I$, $P_\lambda P_\mu = \delta_{\lambda\mu} P_\lambda$.

Worked Example — Spectral Decomposition of X

Diagonalise X

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \det(X - \lambda I) = \lambda^2 - 1 = 0.$$

Eigenvalues $\lambda = \pm 1$ with eigenvectors

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \ (+1), \qquad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} \ (-1).$$

Spectral form

$$X = (+1)\, |+\rangle\langle+| \; + \; (-1)\, |-\rangle\langle-| \,.$$

Check:

$$|+\rangle\langle+| = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \; |-\rangle\langle-| = \tfrac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

$$|+\rangle\langle+| - |-\rangle\langle-| = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X. \; \checkmark$$

Functions of Operators

Definition via the spectrum

If $A = \sum_i \lambda_i \, |v_i\rangle\langle v_i|$ is normal and f is defined on the eigenvalues, set

$$\boxed{f(A) = \sum_i f(\lambda_i) \, |v_i\rangle\langle v_i|}$$

i.e. apply f to the eigenvalues, keep the same eigenvectors.

Examples

- $A^2 = \sum_i \lambda_i^2 \, |v_i\rangle\langle v_i|$.
- $A^{-1} = \sum_i \lambda_i^{-1} \, |v_i\rangle\langle v_i|$ (if all $\lambda_i \neq 0$).
- $\sqrt{A} = \sum_i \sqrt{\lambda_i} \, |v_i\rangle\langle v_i|$ (for A positive).

Consistency

For a polynomial f, this agrees with plugging A into the polynomial — so the definition genuinely extends ordinary matrix powers.

The Operator Exponential

Definition

$$e^{A} = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \sum_i e^{\lambda_i} \, |v_i\rangle\langle v_i| \, .$$

Why quantum cares

For a Hermitian Hamiltonian H, time evolution is unitary:

$$U(t) = e^{-iHt/\hbar}, \qquad U^\dagger U = I \, .$$

(If A is Hermitian, e^{iA} is unitary.)

Handy identity (any involution)

If $A^2 = I$ (e.g. X, Y, Z),

$$\boxed{e^{i\theta A} = \cos\theta \, I + i \sin\theta \, A}.$$

This is the "Euler formula" for Pauli operators and underlies single-qubit rotations.

Worked Example — A Function of Z

Compute $e^{i\theta Z}$

$Z = |0\rangle\langle 0| - |1\rangle\langle 1|$, eigenvalues $+1, -1$. Apply $f(\lambda) = e^{i\theta\lambda}$:

$$e^{i\theta Z} = e^{i\theta}\,|0\rangle\langle 0| + e^{-i\theta}\,|1\rangle\langle 1| = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

Cross-check with the involution identity

Since $Z^2 = I$,

$$\begin{aligned} e^{i\theta Z} &= \cos\theta\, I + i\sin\theta\, Z \\ &= \begin{pmatrix} \cos\theta + i\sin\theta & 0 \\ 0 & \cos\theta - i\sin\theta \end{pmatrix}. \end{aligned}$$

Both give the same matrix. ✓

Note

$|e^{\pm i\theta}| = 1$: the eigenvalues sit on the unit circle, confirming $e^{i\theta Z}$ is unitary.

Quick Check — Round 2 (Spectral Theorem & Functions)

[Q1]

State the spectral theorem for a normal operator A.

[Q2]

What is special about the eigenvalues of (a) a Hermitian and (b) a unitary operator?

[Q3]

A is positive with eigenprojectors P_1 (eigenvalue 4) and P_2 (eigenvalue 9). Write $\sqrt{A}$.

[Q4]

Which property of X is used to prove $e^{i\theta X} = \cos\theta\, I + i\sin\theta\, X$?

Answers — Round 2

[A1]

A normal A has an orthonormal eigenbasis $\{|v_i\rangle\}$ and $A = \sum_i \lambda_i \, |v_i\rangle\langle v_i|$ (equivalently $U^\dagger A U$ is diagonal for some unitary U).

[A2]

(a) Hermitian $\Rightarrow$ eigenvalues are **real**; (b) unitary $\Rightarrow$ eigenvalues have **modulus** 1 ($\lambda = e^{i\varphi}$).

[A3]

$\sqrt{A} = \sqrt{4}\, P_1 + \sqrt{9}\, P_2 = 2P_1 + 3P_2$.

[A4]

The involution property $X^2 = I$ — it collapses the power series into even (cos) and odd (sin) parts.

The Tensor Product

Combining systems

Two systems A, B join as $\mathcal{H}_A \otimes \mathcal{H}_B$, with

$$\dim(\mathcal{H}_A \otimes \mathcal{H}_B) = \dim \mathcal{H}_A \cdot \dim \mathcal{H}_B .$$

Basis: $\{|i\rangle_A \otimes |j\rangle_B\}$, also written $|i\rangle\,|j\rangle$ or $|ij\rangle$.

Kronecker product (vectors)

$$|0\rangle \otimes |1\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} = |01\rangle .$$

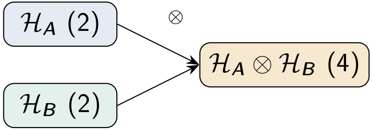

Figure: dimensions multiply under $\otimes$.

Properties and Operators on Tensor Products

Bilinearity

$$(a\,|\psi\rangle) \otimes |\phi\rangle = a(|\psi\rangle \otimes |\phi\rangle) = |\psi\rangle \otimes (a\,|\phi\rangle),$$
$$(|\psi_1\rangle + |\psi_2\rangle) \otimes |\phi\rangle = |\psi_1\rangle \otimes |\phi\rangle + |\psi_2\rangle \otimes |\phi\rangle\,.$$

Inner product factorises

$$\big(\langle\psi_1|\otimes\langle\phi_1|\big)\big(|\psi_2\rangle\otimes|\phi_2\rangle\big) = \langle\psi_1|\psi_2\rangle\ \langle\phi_1|\phi_2\rangle\,.$$

Operators

$$(A \otimes B)(|\psi\rangle \otimes |\phi\rangle) = A\,|\psi\rangle \otimes B\,|\phi\rangle\,,$$
$$(A\otimes B)(C\otimes D) = AC\otimes BD, \quad (A\otimes B)^\dagger = A^\dagger\otimes B^\dagger.$$

Matrix Kronecker product

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{pmatrix}.$$

Worked Example — Kronecker Products

A product state

$$|+\rangle \otimes |0\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle + |10\rangle).$$

As a vector: $\frac{1}{\sqrt{2}}(1,0,1,0)^\top$.

An operator: $Z \otimes X$

$$Z \otimes X = \begin{pmatrix} 1\cdot X & 0 \\ 0 & -1\cdot X \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Sanity check

$\dim = 2 \times 2 = 4$, so $Z \otimes X$ is 4×4, acting on two qubits.

Product States vs. Entangled States

Product (separable)

Can be written as a single tensor product:

$$|\Psi\rangle = |a\rangle \otimes |b\rangle .$$

e.g. $\frac{1}{\sqrt{2}}(|00\rangle + |01\rangle) = |0\rangle \otimes \frac{|0\rangle+|1\rangle}{\sqrt{2}}$.

Entangled

Cannot be factored. The Bell state

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

has no $|a\rangle \otimes |b\rangle$ form.

How to tell them apart cleanly?

The **Schmidt decomposition** gives a basis-independent test: count the nonzero Schmidt coefficients (the *Schmidt rank*).

The Schmidt Decomposition Theorem

Statement

For *any* pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ there exist orthonormal sets $\{|i_A\rangle\} \subset \mathcal{H}_A$, $\{|i_B\rangle\} \subset \mathcal{H}_B$ and reals $\lambda_i \geq 0$ with $\sum_i \lambda_i^2 = 1$ such that

$$\boxed{|\psi\rangle = \sum_i \lambda_i \, |i_A\rangle \otimes |i_B\rangle}$$

The λ_i are the **Schmidt coefficients**; the number of nonzero λ_i is the **Schmidt rank** (Schmidt number).

Entangled test

Entanglement test

$$\text{Schmidt rank} = 1 \iff \text{product state},$$
$$\text{Schmidt rank} > 1 \iff \text{entangled}.$$

Where the numbers come from

The λ_i^2 are the eigenvalues of the reduced states $\rho_A = \mathrm{Tr}_B\, |\psi\rangle\langle\psi|$ and ρ_B; both share the *same* spectrum.

Worked Example — Schmidt Decomposition

Bell state (entangled)

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}|0\rangle|0\rangle + \frac{1}{\sqrt{2}}|1\rangle|1\rangle.$$

Already in Schmidt form: orthonormal $\{|0\rangle, |1\rangle\}$ on both sides, $\lambda_1 = \lambda_2 = \frac{1}{\sqrt{2}}$.

$$\sum_i \lambda_i^2 = \tfrac{1}{2} + \tfrac{1}{2} = 1, \quad \text{Schmidt rank} = \mathbf{2}.$$

A product state

$$|\chi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle) = |0\rangle \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}} = |0\rangle \otimes |+\rangle.$$

Single term $\Rightarrow \lambda_1 = 1$, Schmidt rank $= \mathbf{1} \Rightarrow$ *not* entangled.

Practical recipe

Write amplitudes as a matrix $C = (c_{ij})$ where $|\psi\rangle = \sum_{ij} c_{ij}|i\rangle|j\rangle$; its **singular values** are the Schmidt coefficients (SVD).

Why the Schmidt Decomposition Matters

What it buys us

- A **basis-independent** measure of entanglement (the rank / coefficients).
- Both subsystems have reduced states with the *same* eigenvalues.
- Foundation for *purification* and entanglement measures (e.g. entropy $-\sum \lambda_i^2 \log \lambda_i^2$).

Used throughout QC

Quantum communication, entanglement distillation, tensor-network simulation, and analysing the power of quantum algorithms all lean on Schmidt structure.

Caveat

The Schmidt decomposition is stated for *pure, bipartite* states; multipartite or mixed states need more machinery (later units).

Quick Check — Round 3 (Tensor & Schmidt)

[Q1]

Compute $|1\rangle \otimes |0\rangle$ and state the dimension of the resulting space.

[Q2]

Is $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ a product state? Give its Schmidt rank.

[Q3]

State the Schmidt decomposition theorem (form and constraint on the coefficients).

[Q4] MCQ

Schmidt rank 1 means the state is:
(a) entangled (b) product / separable (c) mixed (d) maximally entangled

Answers — Round 3

[A1]

$|1\rangle \otimes |0\rangle = |10\rangle = (0,0,1,0)^\top$, in a space of dimension $2^2 =$ **4**.

[A2]

Not a product state; it is already in Schmidt form with $\lambda_1 = \lambda_2 = \frac{1}{\sqrt{2}}$, so Schmidt rank $=$ **2** (entangled).

[A3]

$|\psi\rangle = \sum_i \lambda_i \; |i_A\rangle \otimes |i_B\rangle$ with orthonormal $\{|i_A\rangle\}, \{|i_B\rangle\}$, $\lambda_i \geq 0$ and $\sum_i \lambda_i^2 = 1$.

[A4]

(b) product / separable — a single Schmidt term factorises as $|a\rangle \otimes |b\rangle$.

Summary of Unit 2

What we covered

1. **Dirac notation** — kets, bras, inner products $\langle\phi|\psi\rangle$, outer products $|\phi\rangle\langle\psi|$.
2. **Hilbert space** — complex inner-product space; orthonormal bases and completeness $\sum_i |i\rangle\langle i| = I$.
3. **Dual vectors** — bras as functionals; Riesz correspondence; the adjoint $A^\dagger = (A^*)^\top$.
4. **Operator classes** — Hermitian, unitary, normal, positive, projectors.
5. **Spectral theorem** — normal $A = \sum_i \lambda_i |v_i\rangle\langle v_i|$; real / unit-circle eigenvalues.
6. **Functions of operators** — $f(A) = \sum_i f(\lambda_i) |v_i\rangle\langle v_i|$; the exponential $e^{-iHt/\hbar}$.
7. **Tensor products** — Kronecker products; dimensions multiply.
8. **Schmidt decomposition** — $|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle$; rank detects entanglement.

Dirac notation turns quantum mechanics into clean linear algebra.

Practice Questions (Exam Oriented)

Short answer (2–5 marks)

1. Define a Hilbert space and list the inner-product axioms.
2. Explain kets, bras and the ket→bra (adjoint) correspondence with an example.
3. Define Hermitian, unitary and normal operators; give one relation among them.
4. State the spectral theorem and the eigenvalue property for Hermitian and unitary operators.
5. Define the Schmidt rank and state what rank 1 implies.

Long answer (8–10 marks)

6. Prove $(AB)^\dagger = B^\dagger A^\dagger$ and $\big(|\phi\rangle\langle\psi|\big)^\dagger = |\psi\rangle\langle\phi|$.
7. Find the spectral decomposition of X (or Y) and use it to compute $e^{i\theta X}$.
8. Compute $Z \otimes X$ and $H \otimes H$ as 4×4 matrices; apply $H \otimes H$ to $|00\rangle$.
9. State and illustrate the Schmidt decomposition; classify $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and $|0\rangle \otimes |+\rangle$.
10. Show a unitary preserves inner products and maps orthonormal bases to orthonormal bases.

Think & Explore (Take-Home)

Mini task (by hand)

- Find the spectral decomposition of Pauli Y and of the Hadamard H.
- Use it to compute $e^{i\theta Y}$ and verify $e^{i\theta Y} = \cos\theta\, I + i\sin\theta\, Y$.
- Give the Schmidt decomposition of $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$ and classify it.

Coding (self-learning)

In `NumPy`: diagonalise with `eigh`, exponentiate with `scipy.linalg.expm`, tensor with `np.kron`, and get Schmidt coefficients from `np.linalg.svd`.

Reading & reflection

- Nielsen & Chuang, §2.1 (linear algebra for QC).
- Kaye–Laflamme–Mosca, Ch. 2 background.
- Prepare for **Unit 3**: density operators, measurement, and quantum dynamics.

Reminder

Assignment 1 covers Units 1 & 2 — handwritten, step-by-step, submitted as a single PDF via the course page.

References

Textbooks (as per the course page)

1. P. Kaye, R. Laflamme & M. Mosca, *An Introduction to Quantum Computing*, Oxford Univ. Press (Ch. 2). **[Main text]**
2. M. A. Nielsen & I. L. Chuang, *Quantum Computation and Quantum Information*, Cambridge (§2.1).
3. S. Axler, *Linear Algebra Done Right*, Springer (spectral theorem).

Classic & seminal

4. P. A. M. Dirac, *The Principles of Quantum Mechanics*, Oxford — origin of bra–ket notation.
5. P. A. M. Dirac, "A new notation for quantum mechanics," *Math. Proc. Camb. Phil. Soc.*, 1939.
6. E. Schmidt, "Zur Theorie der linearen und nichtlinearen Integralgleichungen," *Math. Ann.*, 1907 — the decomposition.

NPTEL: *Quantum Computing* · Course page: https://www.gcjana.in/courses/shardauniversity/2501/quantum-computing/

Thank You

Any Questions?

"In mathematics you don't understand things. You just get used to them."

— attributed to John von Neumann

Post your doubts on the course page $\rightarrow$
https://www.gcjana.in/courses/shardauniversity/2501/quantum-computing/#Post-doubts