# Fuzzy Logic: From Sets to Systems

An Introduction to Fuzzy Logic Concept

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Soft Computing — Fuzzy Logic Module

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**Course Page:**
https://www.gcjana.in/courses/shardauniversity/2502/PHD650

# Outline

# What is a Set?

## Definition

A **set** is a well-defined collection of distinct objects.

## Examples

- Set of students in a class
- Set of even numbers
- Set of cities in India

## Key Idea

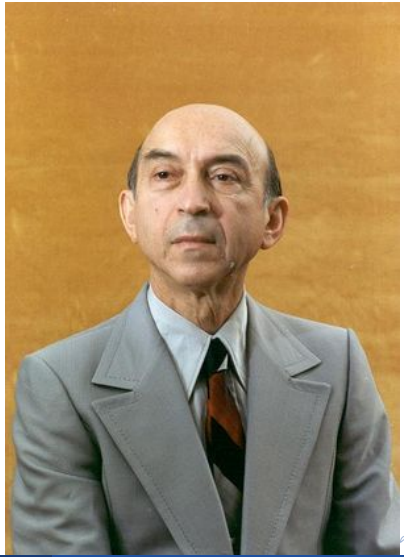
Membership is usually **binary** (belongs or does not belong)

# A Brief History of Fuzzy Logic

- First introduced by **Lotfi A. Zadeh** in 1965
- University of California, Berkeley, USA
- Considered the **father of Fuzzy Logic**

**Interesting Fact**

He is fondly nicknamed as **LAZ**



# Why Can't Computers Do Everything?

## Limitations of Classical Computing

- Algorithms assume **exact**, complete information
- Cannot handle **partial** or **approximate** knowledge
- Real-world information is inherently *vague*

## Typical Problem Areas

- Hospital information management
- Public administration systems
- Flood / fire control systems

## The Solution: Soft Computing

*"Exploit the tolerance for imprecision, uncertainty, partial truth, and approximation to achieve tractability, robustness and low solution cost."*

Soft Computing umbrella:

- **Fuzzy Logic**
- Neural Networks
- Genetic Algorithms
- Chaos Theory

# Crisp vs. Fuzzy Answers

## Crisp Logic

**Question:** Is the liquid colourless?

| Liquid | Answer |
|---|---|
| Water | **Yes** |
| Milk | **No** |
| Coca-Cola | **No** |
| Sprite | **Yes** |

Only two values: **True / False**

## Fuzzy Logic

**Question:** Is the person honest?

| Person | Degree |
|---|---|
| Ankit | 0.99 (*Extremely*) |
| Rajesh | 0.75 (*Very honest*) |
| Santosh | 0.55 (*Honest at times*) |
| Kabita | 0.35 (*Dishonest*) |

Continuum of truth $\in [0, 1]$

**The World is Fuzzy!**

# Crisp Sets — Recap

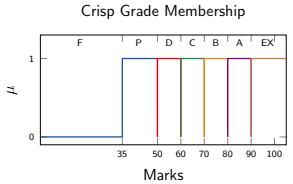
## Definition

A **crisp set** $S \subseteq X$ is a collection of elements where membership is binary: an element either belongs or does not belong.

$$\mu_S(x) = \begin{cases} 1 & x \in S \\ 0 & x \notin S \end{cases}$$

## Example — Course Grades (Crisp)

| | |
|---|---|
| EX | Marks $\geq 90$ |
| A | $80 \leq$ Marks $< 90$ |
| B | $70 \leq$ Marks $< 80$ |
| C | $60 \leq$ Marks $< 70$ |
| D | $50 \leq$ Marks $< 60$ |
| P | $35 \leq$ Marks $< 50$ |
| F | Marks $< 35$ |




Sharp boundaries — a student scoring 79 is B; scoring 80 is instantly A.

# What is a Fuzzy Set?

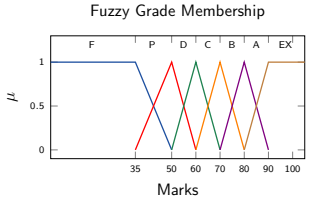
## Formal Definition

A **fuzzy set** $A$ in a universe of discourse $X$ is defined as a set of ordered pairs:

$$A = \{(x, \mu_A(x)) \mid x \in X\}$$

where $\mu_A : X \to [0, 1]$ is the **membership function** (MF). $\mu_A(x)$ is the *grade of membership* of $x$ in $A$.

- $\mu_A(x) = 1 \Rightarrow x$ *fully included* in $A$
- $\mu_A(x) = 0 \Rightarrow x$ *not included* in $A$
- $0 < \mu_A(x) < 1 \Rightarrow$ *partial membership*




## Example

$X =$ cities in India, $A =$ "City of Comfort"

$A = \{$(Delhi, 0.7), (Bangalore, 0.9)

Overlapping grades — a student of 79 can be both B and A

# Crisp Set vs. Fuzzy Set — Comparison

| **Crisp Set** | **Fuzzy Set** |
|---|---|
| $S = \{s \mid s \in X\}$ | $F = \{(s, \mu(s)) \mid s \in X\}$ |
| Collection of elements | Collection of *ordered pairs* |
| Membership: strict Yes / No | Membership: degree $\mu(s) \in [0, 1]$ |
| Boundaries are **sharp** | Boundaries are **gradual** |

**Key Note**

A crisp set *is* a special fuzzy set (membership values are only 0 or 1). A fuzzy set is **not necessarily** a crisp set.

# Fuzzy Set: Discrete and Continuous Universes

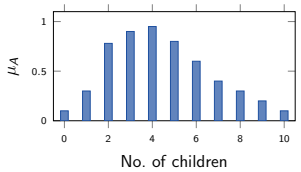
## Discrete Universe

$X = \{0, 1, \ldots, 10\}$ (number of children)

$A =$ "Happy Family"

$A =$

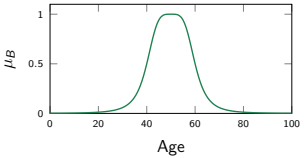
$\{(0, 0.1), (1, 0.30), (2, 0.78), \ldots, (10, 0.1)\}$




## Continuous Universe

$X = \mathbb{R}^+$ (age), $B =$ "Middle Aged"

$$\mu_B(x) = \frac{1}{1 + \left(\frac{x-50}{10}\right)^4}$$



# Fuzzy Set Terminologies

## Support

$\text{supp}(A) = \{x \in X \mid \mu_A(x) > 0\}$

## Core

$\text{core}(A) = \{x \in X \mid \mu_A(x) = 1\}$

## Height

$h(A) = \max_{x \in X} \mu_A(x)$

## Normality

$A$ is *normal* if $\text{core}(A) \neq \emptyset$, i.e. $h(A) = 1$.
Otherwise $A$ is *subnormal*.

## $\alpha$-Cut

$A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}$
*Strong* $\alpha$-cut: $\mu_A(x) > \alpha$
Note: $\text{supp}(A) = A_{0^+}$, $\text{core}(A) = A_1$

## Crossover Point

$\{x \mid \mu_A(x) = 0.5\}$

## Bandwidth

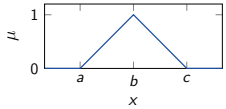
For a normal, convex fuzzy set:
$\text{bw}(A) = |x_1 - x_2|$
where $\mu_A(x_1) = \mu_A(x_2) = 0.5$

## Symmetry

# Standard Membership Functions

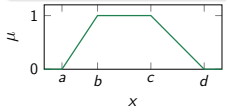
## Triangular MF — $\{a, b, c\}$

$$\mu(x) = \begin{cases} 0 & x \leq a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ \frac{c-x}{c-b} & b \leq x \leq c \\ 0 & c \leq x \end{cases}$$




## Trapezoidal MF — $\{a, b, c, d\}$

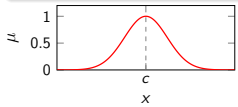
$$\mu(x) = \begin{cases} 0 & x \leq a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & b \leq x \leq c \\ \frac{d-x}{d-c} & c \leq x \leq d \\ 0 & d \leq x \end{cases}$$



# More Membership Functions

## Gaussian MF — $\{c, \sigma\}$

$$\mu(x) = \exp\left[-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2\right]$$




## Generalised Bell MF — $\{a, b, c\}$

$$\mu(x) = \frac{1}{1 + \left|\frac{x-c}{a}\right|^{2b}}$$

Also called *Cauchy MF*.

## Sigmoidal MF — $\{a, c\}$

$$\mu(x) = \frac{1}{1 + e^{-a(x-c)}}$$

$c$ = crossover point, $a$ = slope at $c$

# Operations on Fuzzy Sets

## Complement

$\mu_{\bar{A}}(x) = 1 - \mu_A(x)$

## Union

$\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$

## Intersection

$\mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$

## Bounded Sum

$\mu_{A+B}(x) = \min\{1, \mu_A(x) + \mu_B(x)\}$

## Bounded Difference

$\mu_{A-B}(x) = \max\{0, \mu_A(x) - \mu_B(x)\}$

## Concentration & Dilation

$\mu_{A^\alpha}(x) = [\mu_A(x)]^\alpha$

- $\alpha > 1 \Rightarrow$ *Concentration* ("Very ...")
- $\alpha < 1 \Rightarrow$ *Dilation* ("More or less ...")

## Cartesian Product

$\mu_{A \times B}(x, y) = \min\{\mu_A(x), \mu_B(y)\}$

## Important Difference from Crisp Sets

$A \cap \bar{A}$ is **not** necessarily $\emptyset$

(Law of excluded middle does *not* hold!)

# Properties of Fuzzy Sets

## Commutativity

$A \cup B = B \cup A$
$A \cap B = B \cap A$

## Associativity

$A \cup (B \cup C) = (A \cup B) \cup C$
$A \cap (B \cap C) = (A \cap B) \cap C$

## Distributivity

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

## Idempotence

$A \cup A = A, \quad A \cap A = A$

## Involution

$(\bar{A}^c)^c = A$

## De Morgan's Laws

$(A \cap B)^c = A^c \cup B^c$
$(A \cup B)^c = A^c \cap B^c$

## Transitivity

$A \subseteq B,\ B \subseteq C \ \Rightarrow\ A \subseteq C$

# What is Fuzzy Logic?

## Definition

**Fuzzy Logic** is a form of multi-valued logic derived from fuzzy set theory to deal with reasoning that is *approximate rather than precise*.

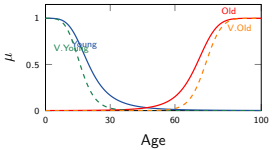
- Degree of truth $\in [0, 1]$, not just {True, False}
- Not "vague logic" — it is logic *for dealing with vague concepts*

## Linguistic Variable

A variable whose values are **words or sentences** in a natural or artificial language.

**Example:** age

$T(\text{age}) = \{\text{young, old, very young, not young, very old}, \ldots\}$



# Fuzzy Conditional Statements & Algorithms

## Fuzzy Conditional Statement

Expressions of the form **"If** $A$ **THEN** $B$**"** where $A$ and $B$ have fuzzy meaning.

*Example:* "If $x$ is *small* THEN $y$ is *large*"

## Fuzzy Algorithm

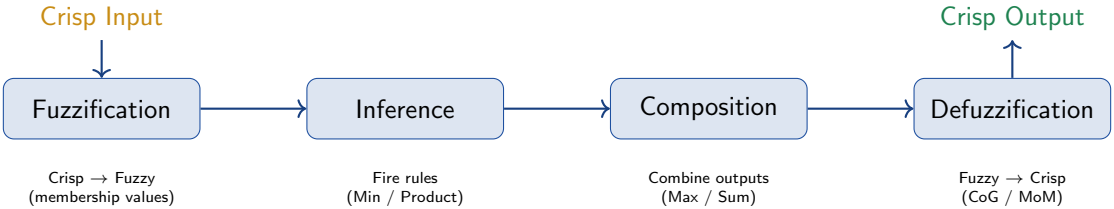
An ordered sequence of instructions which may contain fuzzy assignment and conditional statements.

Execution is governed by the **compositional rule of inference**.

## Practical Application: Fuzzy Washing Machine

- Detects type & amount of laundry $\rightarrow$ adjusts water level
- Foam detection $\rightarrow$ adds extra rinse cycle
- Imbalance compensation $\rightarrow$ calculates max safe spin speed

# The Fuzzy Inference Pipeline




## Inference Methods

**Min** — clips output MF at the truth degree
**Product** — scales output MF by truth degree

## Composition Methods

**MAX** — pointwise maximum over all outputs
**SUM** — pointwise sum (values may exceed 1)

# Defuzzification Techniques

## Centre of Gravity (CoG)

Also called Centre of Area (COA):

$$\eta = \frac{\sum_{y \in Y} y\,\mu(y)}{\sum_{y \in Y} \mu(y)}$$

Considers rules proportionally by their degree of applicability.

## Mean of Maximum (MoM)

Applies an additional $\alpha$-level cut at the maximum of the output fuzzy set, then averages.

## Note

CoG and MoM give **different** crisp output values. CoG is smoother; MoM can be more conservative.

## Air Conditioner Example ($T = 16°C$)

**Fuzzify:** $\mu_{\text{Cool}} = 0.3$, $\mu_{\text{Pleasant}} = 0.4$

**Infer:** Rules "Cool→Slow" & "Pleasant→Medium" fire

**Compose:** $\alpha$-level cuts at 0.3 (Slow) and 0.4 (Medium)

# Conventional vs. Fuzzy Control

## Conventional (PID) Controller

$$u(t) = K_P e(t) + K_I \int_0^t e(\tau)\, d\tau + K_D \frac{de}{dt}$$

| | |
|---|---|
| $K_P$ | current error |
| $K_I$ | sum of recent errors |
| $K_D$ | rate of error change |

Requires a **precise mathematical model**. Fails when model is unknown, non-linear, or hard to obtain.

## Fuzzy Controller

Maps input linguistic variables → output linguistic variables via a **rule base**.

Two main types:

- **Mamdani** (linguistic) — direct closed-loop
- **Takagi-Sugeno-Kang** — supervisory

Modes: *Feedback* (control) or *Feed-forward* (prediction).

## Advantage

Encodes **human expert knowledge** in linguistic rules, no explicit model required.

# Fuzzy Logic vs. Probability

## Fuzzy (Certainty / Vagueness)

**"The patient probably has flu."**

Doctor assigns certainty 60%.
Disease category itself is vague.
Captures *degree of belonging*.

## Probability (Chance of Event)

**"India will win with 60% chance."**

Out of 100 matches, India wins 60.
Event outcome is crisp (win / loss).
Captures *frequency of occurrence*.

| **Fuzzy Logic** | **Probability Theory** |
|---|---|
| Based on best guess from experience | Based on recorded past data |
| Models vagueness / imprecision | Models uncertainty / randomness |
| *Prediction* | *Forecasting* |

# Summary

## What We Covered

1. **Crisp Sets** — binary membership, sharp boundaries
2. **Fuzzy Sets** — graded membership $\mu(x) \in [0, 1]$
3. **Key Terminologies** — support, core, $\alpha$-cut, normality, bandwidth
4. **Membership Functions** — triangular, trapezoidal, Gaussian, bell, sigmoid
5. **Operations** — complement, union, intersection, concentration, dilation
6. **Linguistic Variables** — word-based values with semantic rules
7. **Fuzzy Inference Pipeline** — fuzzify $\rightarrow$ infer $\rightarrow$ compose $\rightarrow$ defuzzify
8. **Defuzzification** — CoG and MoM methods
9. **Fuzzy Control** — rule-based controllers without explicit plant model
10. **Fuzzy vs. Probability** — vagueness vs. randomness

**Fuzzy Logic gives machines the power to reason like humans.**

# Thank You

Any Questions?

*"The closer you look at reality, the fuzzier it gets."*
— Lotfi A. Zadeh (LAZ)