Fuzzy Sets — Basic Terminology

Class Task: The Fuzzy Set “Young Age”

Classroom Activity

24 July 2026

What we will cover

Example 1: The Fuzzy Set "Young"

Universe of discourse:

$$X = [0, 80] \text{ years}$$

We define the fuzzy set **Young** using a **trapezoidal** membership function.

Age (years)	μ_{Young}**(x)**
0	1.0
10	1.0
20	1.0
30	0.5
40	0
50	0
60	0
80	0

Key idea

In a *crisp* set, an age is either young or not young (0 or 1).
In a *fuzzy* set, an age can be young *to a degree* — any value in $[0, 1]$.

Ask the class

Is a 30-year-old young?
Answer: half young, half not $\Rightarrow \mu = 0.5$.

The Membership Function in Closed Form

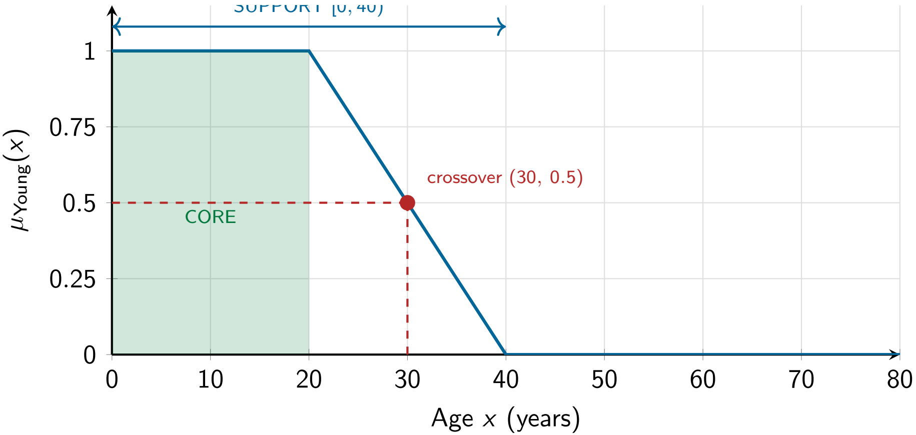
Reading the table, the shape is **flat then falling** — a left-shouldered trapezoid:

$$\mu_{\text{Young}}(x) = \begin{cases} 1, & 0 \leq x \leq 20 \\ \dfrac{40 - x}{20}, & 20 < x \leq 40 \\ 0, & 40 < x \leq 80 \end{cases}$$

Verify with the table

- $\mu_{\text{Young}}(20) = 1 \quad \checkmark$
- $\mu_{\text{Young}}(30) = \frac{40-30}{20} = \frac{10}{20} = 0.5 \quad \checkmark$
- $\mu_{\text{Young}}(40) = \frac{40-40}{20} = 0 \quad \checkmark$
- $\mu_{\text{Young}}(25) = \frac{15}{20} = 0.75, \quad \mu_{\text{Young}}(35) = \frac{5}{20} = 0.25$

The Graph — Everything Can Be Read From Here



Flat at the top up to age 20, then falls linearly to zero at age 40.

1. Support

Definition

$$\operatorname{supp}(A) = \{\, x \in X \mid \mu_A(x) > 0 \,\}$$

The set of all elements that belong to A *even slightly*.

Ask: Which ages have membership *greater than zero*?

- 0 yrs $\rightarrow \mu = 1.0$ ✓
- 10 yrs $\rightarrow \mu = 1.0$ ✓
- 20 yrs $\rightarrow \mu = 1.0$ ✓
- 25 yrs $\rightarrow \mu = 0.75$ ✓
- 35 yrs $\rightarrow \mu = 0.25$ ✓
- 40 yrs $\rightarrow \mu = 0$ ×

Answer

$$\operatorname{supp}(\text{Young}) = [0, 40)$$

Meaning: these are all the ages that are considered *young to some extent*.

Careful — why $[0, 40)$ and not $[0, 40]$?

The definition demands $\mu > 0$ *strictly*. At exactly $x = 40$ we have $\mu = 0$, so 40 is **excluded**. The bracket is open at 40.

2. Core

Definition

$$\mathrm{core}(A) = \{\, x \in X \mid \mu_A(x) = 1 \,\}$$

The elements that belong to A *completely*.

Ask: Which ages are *fully* young?

Answer

Membership equals 1 for every age from 0 to 20:

$$\mathrm{core}(\mathrm{Young}) = [0, 20]$$

Meaning: these ages are **100% young** — no doubt at all.

Bonus term: Boundary

$$\mathrm{bd}(A) = \{\, x \mid 0 < \mu_A(x) < 1 \,\} = \mathrm{supp}(A) \setminus \mathrm{core}(A)$$

Here: $\mathrm{bd}(\mathrm{Young}) = (20, 40)$ — the *fuzzy*, uncertain region.

3. Height

Definition

$$h(A) = \sup_{x \in X} \mu_A(x) \quad \text{(the maximum membership reached)}$$

Ask: What is the tallest point of the graph?

Answer

The curve reaches 1 on the flat top, and never goes higher.

$$h(\text{Young}) = 1$$

Note

Height tells you *how strongly* the best element belongs. It says nothing about *how many* elements belong — that is cardinality.

4. Normal vs. Subnormal Fuzzy Set

Definition

A is **normal** $\iff h(A) = 1$ $\qquad$ A is **subnormal** $\iff h(A) < 1$

Equivalently: A is normal $\iff \text{core}(A) \neq \emptyset$.

Answer

Since $h(\text{Young}) = 1$, our set is a **Normal Fuzzy Set**.

Now ask: Suppose the highest membership were only 0.8. Would it still be normal?

Students should answer: No

It becomes **subnormal**. Its core would be *empty* — nobody is fully young.

Normalisation: we can rescale it back to normal by

$$\mu_{A'}(x) = \frac{\mu_A(x)}{h(A)} = \frac{\mu_A(x)}{0.8}$$

5. Crossover Point

Definition

A point x where

$$\mu_A(x) = 0.5$$

the age of *maximum ambiguity* — equally young and not young.

Ask: Where does the graph cut the 0.5 line?

Answer

Solve $\dfrac{40 - x}{20} = 0.5 \Rightarrow 40 - x = 10 \Rightarrow x = 30.$

$$\text{Crossover point} = 30 \text{ years}$$

Discussion

A fuzzy set may have **0, 1, or 2** crossover points.

- Our set has **one**, because the left side is flat (a shoulder) and never crosses 0.5.
- A *symmetric* triangle or trapezoid has **two**.

6. α-cut (Alpha Cut)

Definition

$$A_\alpha = \{\, x \in X \mid \mu_A(x) \geq \alpha \,\}$$

An α-cut converts a fuzzy set into a **crisp set** at threshold α.

Take $\alpha = 0.5$. Which ages survive?

- $0 \to 1.0 \geq 0.5$ ✓
- $10 \to 1.0 \geq 0.5$ ✓
- $20 \to 1.0 \geq 0.5$ ✓
- $25 \to 0.75 \geq 0.5$ ✓
- $30 \to 0.5 \geq 0.5$ ✓ (*equality counts!*)
- $35 \to 0.25 < 0.5$ ×

Answer

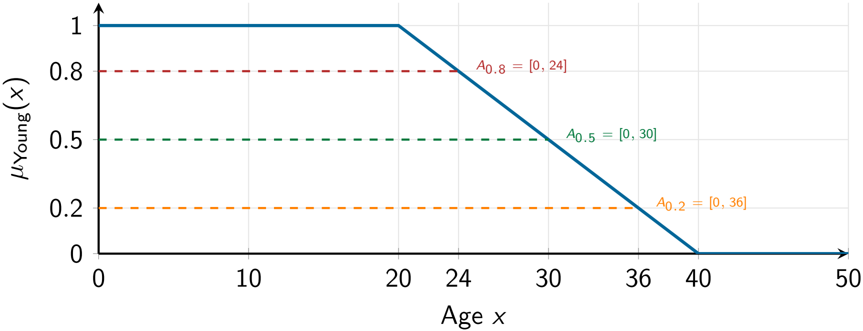
$$A_{0.5} = [0, 30]$$

Now ask: what happens if $\alpha = 0.8$?

$$\frac{40 - x}{20} \geq 0.8 \Rightarrow 40 - x \geq 16 \Rightarrow x \leq 24 \quad \Longrightarrow \quad A_{0.8} = [0, 24]$$

Higher α $\Rightarrow$ stricter selection $\Rightarrow$ smaller interval.

α-cuts are Nested



Nesting property

$$\alpha_1 < \alpha_2 \implies A_{\alpha_2} \subseteq A_{\alpha_1}$$

$$A_{0.8} \subseteq A_{0.5} \subseteq A_{0.2} \subseteq \text{supp}(A)$$

Also: $A_1 = \text{core}(A)$ and $A_{0^+} = \text{supp}(A)$.

7. Strong α-cut

Definition

$$A_\alpha^+ = \{\, x \in X \mid \mu_A(x) > \alpha \,\} \qquad \text{(strictly greater)}$$

Take $\alpha = 0.5$ again. Age 30 has membership *exactly* 0.5. Strictly greater than 0.5? No. So 30 is **removed**.

Answer

$$A_{0.5}^+ = [0, 30)$$

	Normal α-cut	**Strong α-cut**
Condition	$\mu(x) \geq \alpha$	$\mu(x) > \alpha$
At $\alpha = 0.5$	$[0, 30]$	$[0, 30)$
Boundary point	included	excluded
Type of interval	closed	half-open

One-line summary

The two differ *only* at points where $\mu(x)$ equals α exactly.

8. Bandwidth

Definition

For a *normal and convex* fuzzy set with two crossover points $x_1 < x_2$:

$$\mathrm{bw}(A) = |x_2 - x_1|$$

It measures the **width of the ambiguous zone** — how wide the "half-belonging" band is.

Answer

Our set has **only one** crossover point ($x = 30$), because the left side is a flat shoulder that never descends to 0.5.

Therefore bandwidth is **not meaningfully defined** for this particular set. (Some texts take it as 30, measuring from the left edge of the universe — but strictly, two crossovers are required.)

Fix it

Change "Young" to a symmetric triangle around age 25 and two crossover points appear — then bandwidth becomes meaningful. See Practice Q1 and Q3.

Extra Terms Worth Knowing

Cardinality (scalar / sigma count)

$$|A| = \sum_{x \in X} \mu_A(x) \quad \text{(discrete)} \qquad |A| = \int_X \mu_A(x)\, dx \quad \text{(continuous)}$$

Using only the eight tabulated ages: $|A| = 1 + 1 + 1 + 0.5 + 0 + 0 + 0 + 0 = \mathbf{3.5}$

Over the continuum: $|A| = \int_0^{20} 1\, dx + \int_{20}^{40} \frac{40-x}{20}\, dx = 20 + 10 = \mathbf{30}$

Convex fuzzy set

$\mu(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu(x_1), \mu(x_2))$

Equivalently: *every* α-cut is an interval. Our set: convex ✓

Fuzzy singleton

A fuzzy set whose support is a single point:

$$\mu(x) = 1 \text{ at } x = x_0, \quad 0 \text{ elsewhere}$$

Used to fuzzify a crisp input.

Decomposition (Resolution) theorem

Any fuzzy set can be rebuilt from its α-cuts:

$$A = \bigcup_{\alpha \in (0,1]} \alpha \cdot A_\alpha$$

Master Summary — Example 1

Term	Definition	Answer for "Young"
Support	$\mu(x) > 0$	$[0, 40)$
Core	$\mu(x) = 1$	$[0, 20]$
Boundary	$0 < \mu(x) < 1$	$(20, 40)$
Height	$\max \mu(x)$	1
Normal?	$h(A) = 1$	Yes — normal
Crossover	$\mu(x) = 0.5$	$x = 30$
α-cut (α=0.5)	$\mu(x) \geq \alpha$	$[0, 30]$
α-cut (α=0.8)	$\mu(x) \geq \alpha$	$[0, 24]$
Strong α-cut (α=0.5)	$\mu(x) > \alpha$	$[0, 30)$
Bandwidth	$\lvert x_2 - x_1 \rvert$	not defined (one crossover)
Cardinality	$\int \mu(x)\, dx$	30
Convex?	all α-cuts intervals	Yes

Standard Membership Function Models

1. Triangular — $\texttt{trimf}(x; a, b, c)$

$$\mu(x) = \max\left(\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b}\right), 0\right)$$

Core $= \{b\}$, Support $= (a, c)$. Simplest, cheapest, most used in fuzzy control.

2. Trapezoidal — $\texttt{trapmf}(x; a, b, c, d)$

$$\mu(x) = \max\left(\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right), 0\right)$$

Core $= [b, c]$, Support $= (a, d)$. Our "Young" set is a *left-shouldered* trapezoid with $a = b = 0$, $c = 20$, $d = 40$.

3. Gaussian — $\texttt{gaussmf}(x; c, \sigma)$

$$\mu(x) = \exp\left(-\frac{(x-c)^2}{2\sigma^2}\right)$$

Smooth, never exactly zero $\Rightarrow$ support is the whole universe. Core $= \{c\}$.

Standard Membership Function Models (contd.)

4. Generalised Bell — `gbellmf`$(x; a, b, c)$

$$\mu(x) = \frac{1}{1 + \left|\frac{x - c}{a}\right|^{2b}}$$

Nice fact: the crossover points are exactly $c \pm a$, so **bandwidth** $= 2a$. Parameter b controls steepness.

5. Sigmoidal — `sigmf`$(x; a, c)$

$$\mu(x) = \frac{1}{1 + e^{-a(x-c)}}$$

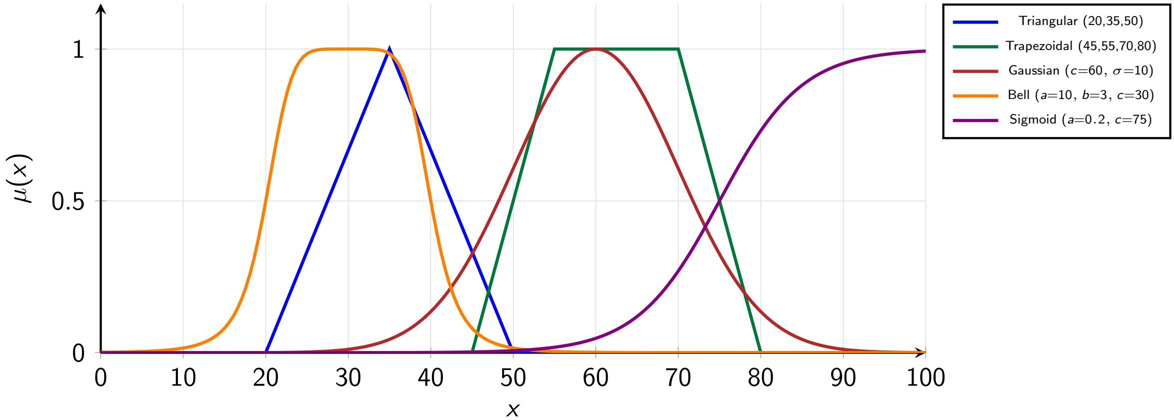
Open on one side — perfect for "very tall", "very hot". Crossover at $x = c$. $a > 0$ opens right, $a < 0$ opens left.

6. Singleton

$$\mu(x) = 1 \text{ if } x = x_0, \quad 0 \text{ otherwise}$$

Used to feed a crisp sensor reading into a fuzzy system.

The Models, Side by Side



Triangular/trapezoidal $\rightarrow$ computationally cheap. Gaussian/bell $\rightarrow$ smooth, differentiable (needed for neuro-fuzzy training).

Practice Question 1 — Triangular "Tall Person"

Question. On $X = [150, 210]$ cm, define **Tall** as triangular with $a = 160$, $b = 180$, $c = 200$:

$$\mu_{\text{Tall}}(x) = \begin{cases} \dfrac{x - 160}{20}, & 160 \le x \le 180 \\ \dfrac{200 - x}{20}, & 180 < x \le 200 \\ 0, & \text{otherwise} \end{cases}$$

Find: support, core, height, normality, crossover points, bandwidth, $A_{0.5}$, $A_{0.5}^{+}$, $A_{0.25}$.

Hint

Now there are *two* sloping sides — so expect **two** crossover points and a real bandwidth.

Practice Question 1 — Triangular "Tall Person"

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Hint

Now there are *two* sloping sides — so expect **two** crossover points and a real bandwidth.

Answer 1 — Worked Out

Support: $\mu > 0$ strictly between the feet

$$\text{supp} = (160, 200)$$

Core: $\mu = 1$ only at the apex

$$\text{core} = \{180\}$$

Height: $h = 1 \Rightarrow$ normal

Crossover points: set $\mu = 0.5$

$$\tfrac{x-160}{20} = 0.5 \Rightarrow x = 170$$

$$\tfrac{200-x}{20} = 0.5 \Rightarrow x = 190$$

Bandwidth: $|190 - 170| = \mathbf{20}$

α-**cut,** $\alpha = 0.5$:

$$A_{0.5} = [170, 190]$$

Strong α-**cut,** $\alpha = 0.5$:

$$A^{+}_{0.5} = (170, 190)$$

both endpoints drop out, since both equal 0.5 exactly.

α-**cut,** $\alpha = 0.25$:

$$\tfrac{x-160}{20} \geq 0.25 \Rightarrow x \geq 165$$

$$\tfrac{200-x}{20} \geq 0.25 \Rightarrow x \leq 195$$

$$A_{0.25} = [165, 195]$$

Contrast with Example 1

Symmetric shape $\Rightarrow$ two crossovers $\Rightarrow$ bandwidth exists. The shoulder in "Young" was what killed it.

Practice Question 2 — A Discrete, Subnormal Set

Question. Let $X = \{1, 2, 3, 4, 5\}$ and

$$B = \{(1,\ 0.2),\ (2,\ 0.6),\ (3,\ 0.8),\ (4,\ 0.5),\ (5,\ 0.1)\}$$

Find support, core, height, normality, crossover, $B_{0.5}$, $B^{+}_{0.5}$, cardinality. Then *normalise* it.

Support: all elements with $\mu > 0$

$$\text{supp}(B) = \{1, 2, 3, 4, 5\}$$

Core: nobody has $\mu = 1$

$$\text{core}(B) = \emptyset$$

Height: $h(B) = \max = 0.8$
Normal? No — subnormal, since $h < 1$.
Crossover: $\mu(4) = 0.5 \Rightarrow x = 4$

$B_{0.5}$ (≥ 0.5): $\{2, 3, 4\}$
$B^{+}_{0.5}$ (> 0.5): $\{2, 3\}$ (4 drops out)
Cardinality:

$$|B| = 0.2 + 0.6 + 0.8 + 0.5 + 0.1 = 2.2$$

Normalised ($\div\, 0.8$):

$$\{(1, 0.25), (2, 0.75), (3, 1), (4, 0.625), (5, 0.125)\}$$

Now $h = 1$ and core $= \{3\}$.

Practice Question 2 — A Discrete, Subnormal Set

Question. Let $X = \{1, 2, 3, 4, 5\}$ and

$$B = \{(1,\ 0.2),\ (2,\ 0.6),\ (3,\ 0.8),\ (4,\ 0.5),\ (5,\ 0.1)\}$$

Find support, core, height, normality, crossover, $B_{0.5}$, $B^{+}_{0.5}$, cardinality. Then *normalise* it.

Support: all elements with $\mu > 0$

$$\text{supp}(B) = \{1, 2, 3, 4, 5\}$$

Core: nobody has $\mu = 1$

$$\text{core}(B) = \emptyset$$

Height: $h(B) = \max = 0.8$
Normal? No — subnormal, since $h < 1$.
Crossover: $\mu(4) = 0.5 \Rightarrow x = 4$

$B_{0.5}$ (≥ 0.5): $\{2, 3, 4\}$
$B^{+}_{0.5}$ (> 0.5): $\{2, 3\}$ (4 drops out)
Cardinality:

$$|B| = 0.2 + 0.6 + 0.8 + 0.5 + 0.1 = 2.2$$

Normalised ($\div\, 0.8$):

$$\{(1, 0.25), (2, 0.75), (3, 1), (4, 0.625), (5, 0.125)\}$$

Now $h = 1$ and core $= \{3\}$.

Practice Question 3 — Trapezoidal "Warm Temperature"

Question. On $X = [0, 50]\,^\circ$C, define **Warm** $=$ `trapmf`$(20, 25, 30, 35)$. Find all the standard quantities, plus $A_{0.8}$.

$$\mu(x) = \begin{cases} \frac{x-20}{5}, & 20 \le x \le 25 \\ 1, & 25 < x \le 30 \\ \frac{35-x}{5}, & 30 < x \le 35 \\ 0, & \text{otherwise} \end{cases}$$

Support $= (20, 35)$
Core $= [25, 30]$
Boundary $= (20, 25) \cup (30, 35)$
Height $= 1 \Rightarrow$ normal
Convex? Yes

Crossovers: $\frac{x-20}{5} = 0.5 \Rightarrow x = 22.5$;
$\frac{35-x}{5} = 0.5 \Rightarrow x = 32.5$
Bandwidth $= 32.5 - 22.5 = 10$
$A_{0.5} = [22.5,\ 32.5]$
$A_{0.5}^{+} = (22.5,\ 32.5)$
$A_{0.8}$: $x \ge 24$ and $x \le 31 \Rightarrow [24, 31]$

Cardinality

Area $=$ two triangles $+$ rectangle $= \frac{1}{2}(5)(1) + (5)(1) + \frac{1}{2}(5)(1) = 10$

Practice Question 3 — Trapezoidal "Warm Temperature"

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Support $= (20, 35)$
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Boundary $= (20, 25) \cup (30, 35)$
Height $= 1 \Rightarrow$ normal
Convex? Yes

Crossovers: $\frac{x-20}{5} = 0.5 \Rightarrow x = 22.5$;
$\frac{35-x}{5} = 0.5 \Rightarrow x = 32.5$
Bandwidth $= 32.5 - 22.5 = \mathbf{10}$
$A_{0.5} = [22.5,\ 32.5]$
$A^{+}_{0.5} = (22.5,\ 32.5)$
$A_{0.8}$: $x \ge 24$ and $x \le 31 \Rightarrow [24, 31]$

Cardinality

Area $=$ two triangles $+$ rectangle $= \frac{1}{2}(5)(1) + (5)(1) + \frac{1}{2}(5)(1) = 10$

Practice Question 4 — Gaussian "Medium Speed"

Question. **Medium** speed on $X = [0, 120]$ km/h is Gaussian with $c = 60$, $\sigma = 10$:

$$\mu(x) = \exp\left(-\frac{(x-60)^2}{200}\right)$$

Find core, height, support, crossover points, bandwidth, and a general formula for A_α.

Core $= \{60\}$ (only the peak reaches 1)
Height $= 1 \Rightarrow$ normal
Support: μ is *never* exactly 0, so support $=$ the whole universe $[0, 120]$.
(In practice we truncate at, say, $\mu < 0.01$.)

Crossovers: set $\mu = 0.5$

$$\frac{(x-60)^2}{200} = \ln 2 \Rightarrow x - 60 = \pm 11.77$$

$$x_1 = 48.23, \qquad x_2 = 71.77$$

Bandwidth $= 23.55$ $(= 2\sigma\sqrt{2\ln 2})$

General α-cut

$$A_\alpha = \left[c - \sigma\sqrt{2\ln(1/\alpha)},\;\; c + \sigma\sqrt{2\ln(1/\alpha)}\right]$$

Check $\alpha = 0.5$: $\sigma\sqrt{2\ln 2} = 10(1.177) = 11.77$ ✓

Practice Question 4 — Gaussian "Medium Speed"

Question. Medium speed on $X = [0, 120]$ km/h is Gaussian with $c = 60$, $\sigma = 10$:

$$\mu(x) = \exp\left(-\frac{(x-60)^2}{200}\right)$$

Find core, height, support, crossover points, bandwidth, and a general formula for A_α.

Core $= \{60\}$ (only the peak reaches 1)
Height $= 1 \Rightarrow$ normal
Support: μ is *never* exactly 0, so support $=$ the whole universe $[0, 120]$.
(In practice we truncate at, say, $\mu < 0.01$*.)*

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$$x_1 = 48.23, \qquad x_2 = 71.77$$

Bandwidth $= 23.55$ $(= 2\sigma\sqrt{2\ln 2})$

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Check $\alpha = 0.5$: $\sigma\sqrt{2\ln 2} = 10(1.177) = 11.77$ ✓

Practice Question 5 — Convexity Test

Question. Is this fuzzy set convex?

$$C = \{(1,\ 0.2),\ (2,\ 0.8),\ (3,\ 0.3),\ (4,\ 0.9),\ (5,\ 0.4)\}$$

Answer

No — it is non-convex.

Test 1 (definition). Take $x_1 = 2$, $x_2 = 4$, midpoint $x = 3$:

$$\mu(3) = 0.3 \ \not\geq \ \min\left(\mu(2), \mu(4)\right) = \min(0.8, 0.9) = 0.8$$

Test 2 (easier — α-cut). Take $\alpha = 0.5$: $C_{0.5} = \{2, 4\}$ — not a contiguous interval, it has a hole at 3. A convex fuzzy set needs *every* α-cut to be a single interval.

Why it matters

The set is *bimodal* — two separate peaks. Linguistic terms like "young" or "tall" should always be convex; non-convexity usually means two concepts got merged by mistake.

Practice Question 5 — Convexity Test

Question. Is this fuzzy set convex?

$$C = \{(1,\ 0.2),\ (2,\ 0.8),\ (3,\ 0.3),\ (4,\ 0.9),\ (5,\ 0.4)\}$$

Answer

No — it is non-convex.

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$$\mu(3) = 0.3 \ \not\geq\ \min\big(\mu(2), \mu(4)\big) = \min(0.8, 0.9) = 0.8$$

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Why it matters

The set is *bimodal* — two separate peaks. Linguistic terms like "young" or "tall" should always be convex; non-convexity usually means two concepts got merged by mistake.

Practice Question 6 — Trick Question on Strong α-cut

Question. For the "Young" set of Example 1, compute $A_{0.75}$ and $A^+_{0.75}$. Are they different? Then compute $A_{0.6}$ and $A^+_{0.6}$. Are *those* different?

$\alpha = 0.75$:

$$\frac{40 - x}{20} \geq 0.75 \Rightarrow x \leq 25$$

$$A_{0.75} = [0, 25], \qquad A^+_{0.75} = [0, 25)$$

Different — because $x = 25$ hits $\mu = 0.75$ *exactly*.

$\alpha = 0.6$:

$$\frac{40 - x}{20} \geq 0.6 \Rightarrow x \leq 28$$

$$A_{0.6} = [0, 28], \qquad A^+_{0.6} = [0, 28)$$

Also different — $\mu(28) = 0.6$ exactly.

Answer

For a *continuous* membership function that actually attains the value α, the normal and strong α-cuts **always** differ — by exactly the boundary point(s).

They coincide only when *no* x satisfies $\mu(x) = \alpha$ — e.g. in a discrete set where α falls between two membership values. Try $\alpha = 0.55$ in Practice Q2: $B_{0.55} = B^+_{0.55} = \{2, 3\}$.

Practice Question 6 — Trick Question on Strong α-cut

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$\alpha = 0.75$:

$$\frac{40 - x}{20} \geq 0.75 \Rightarrow x \leq 25$$

$$A_{0.75} = [0, 25], \qquad A^+_{0.75} = [0, 25)$$

Different — because $x = 25$ hits $\mu = 0.75$ *exactly*.

$\alpha = 0.6$:

$$\frac{40 - x}{20} \geq 0.6 \Rightarrow x \leq 28$$

$$A_{0.6} = [0, 28], \qquad A^+_{0.6} = [0, 28)$$

Also different — $\mu(28) = 0.6$ exactly.

Answer

For a *continuous* membership function that actually attains the value α, the normal and strong α-cuts **always** differ — by exactly the boundary point(s).

They coincide only when *no* x satisfies $\mu(x) = \alpha$ — e.g. in a discrete set where α falls between two membership values. Try $\alpha = 0.55$ in Practice Q2: $B_{0.55} = B^+_{0.55} = \{2, 3\}$.

Practice Question 7 — Reverse Engineering

Question. A normal, convex fuzzy set has support $= (10, 50)$, core $= [20, 40]$, and linear sides. Write its membership function, and find its bandwidth.

Answer

Support edges give the feet: $a = 10$, $d = 50$. Core edges give the shoulders: $b = 20$, $c = 40$. It is $\texttt{trapmf}(10, 20, 40, 50)$:

$$\mu(x) = \begin{cases} \frac{x-10}{10}, & 10 \le x \le 20 \\ 1, & 20 < x \le 40 \\ \frac{50-x}{10}, & 40 < x \le 50 \\ 0, & \text{otherwise} \end{cases}$$

Crossovers: $\frac{x-10}{10} = 0.5 \Rightarrow x = 15$; $\quad \frac{50-x}{10} = 0.5 \Rightarrow x = 45$.
Bandwidth $= 45 - 15 = 30$.

Notice

A trapezoid's bandwidth always equals core width + (half of each slope's base) $= 20 + 5 + 5 = 30$. ✓

Practice Question 7 — Reverse Engineering

Question. A normal, convex fuzzy set has support $= (10, 50)$, core $= [20, 40]$, and linear sides. Write its membership function, and find its bandwidth.

Answer

Support edges give the feet: $a = 10$, $d = 50$. Core edges give the shoulders: $b = 20$, $c = 40$. It is $\texttt{trapmf}(10, 20, 40, 50)$:

$$\mu(x) = \begin{cases} \frac{x-10}{10}, & 10 \leq x \leq 20 \\ 1, & 20 < x \leq 40 \\ \frac{50-x}{10}, & 40 < x \leq 50 \\ 0, & \text{otherwise} \end{cases}$$

Crossovers: $\frac{x-10}{10} = 0.5 \Rightarrow x = 15$; $\quad \frac{50-x}{10} = 0.5 \Rightarrow x = 45$.
Bandwidth $= 45 - 15 = \mathbf{30}$.

Notice

A trapezoid's bandwidth always equals $\;$ core width + (half of each slope's base) $\; = 20 + 5 + 5 = 30.$ ✓

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Common Mistakes to Avoid

1. Open vs. closed brackets

Support uses $\mu > 0$ strictly $\Rightarrow$ endpoints where $\mu = 0$ are **excluded**: $[0, 40)$, not $[0, 40]$.

2. Forgetting equality in the α-cut

A_α uses $\geq$. The point where $\mu = \alpha$ **is included**. Only the *strong* cut throws it away.

3. Assuming two crossover points always exist

Shouldered (flat-edged) and monotone (sigmoid) sets have one — or none. Bandwidth then needs care.

4. Confusing height with cardinality

Height = *how strongly* the best element belongs. Cardinality = *total* amount of membership.

Take-Home Exercise

Define **"Middle-aged"** on $X = [0, 80]$ as $\texttt{trapmf}(30, 40, 55, 65)$.

Compute and hand in:

1. Support, core, boundary
2. Height; is it normal?
3. Both crossover points, and the bandwidth
4. $A_{0.5}$, $A^{+}_{0.5}$, $A_{0.9}$
5. Cardinality (area under the curve)
6. Sketch the graph and mark every quantity on it
7. Now suppose every membership value is multiplied by 0.7. Which of the answers above change, and which stay the same?

Answer check for (3)

Crossovers at $x = 35$ and $x = 60$; bandwidth $= 25$.

Thank You

Questions?